

# EDGE CUT DOMINATION, IRREDUNDANCE, AND INDEPENDENCE IN GRAPHS

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**ABSTRACT.** An edge dominating set  $F$  of a graph  $G = (V, E)$  is an *edge cut dominating set* if the subgraph  $\langle V, G - F \rangle$  is disconnected. The *edge cut domination number*  $\gamma_{ct}(G)$  of  $G$  is the minimum cardinality of an edge cut dominating set of  $G$ . In this paper we study the edge cut domination number and investigate its relationships with other parameters of graphs. We also introduce the properties edge cut irredundance and edge cut independence.

## 1. INTRODUCTION

Let  $G = (V, E)$  be a graph of order  $n = |V|$  and size  $m = |E|$ . Here we often take  $G$  to be a connected simple graph. The *open neighborhood* of a vertex  $v \in V$  is the set  $N(v) = \{u \mid uv \in E\}$ , while the *closed neighborhood* of  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . Similarly, the *closed neighborhood* of a set  $S \subseteq V$  is  $N[S] = \bigcup_{v \in S} N[v]$ .

A *dominating set* is a set  $S \subseteq V$  for which  $N[S] = V$ . The *domination number*  $\gamma(G)$  equals the minimum cardinality over all dominating sets in  $G$ , and a dominating set of cardinality  $\gamma(G)$  is called a  $\gamma$ -set. A dominating set  $S$  is called a *split dominating set* if the induced subgraph  $\langle V - S \rangle$  is either disconnected or  $K_1$ . The *split domination number*  $\gamma_s(G)$  is the minimum cardinality of a split dominating set. This parameter was introduced by Kulli and Janakiram in 1997 [3].

Similar parameters can be defined in terms of a set of edges. A set  $F \subseteq E$  is called an *edge dominating set* if every edge not in  $F$  is adjacent to an edge in  $F$ , that is, has a vertex in common with an edge in  $F$ . The *edge domination number*  $\gamma'(G)$  is the minimum cardinality over all edge dominating sets of  $G$ , and an edge dominating set of cardinality  $\gamma'(G)$  is called a  $\gamma'$ -set. An edge dominating set  $F$  is called an *edge cut dominating set* if the subgraph  $\langle V, E - F \rangle$  is disconnected. The *edge cut domination number*  $\gamma_{ct}(G)$  is the minimum cardinality over all edge cut dominating sets of  $G$ , and an edge cut dominating set of cardinality  $\gamma_{ct}(G)$  is called a  $\gamma_{ct}$ -set.

In 2001, Neeralagi and Nayak first introduced the edge cut domination number [4]; however, they named this parameter the *split edge domination number*. We adopt the term edge cut domination number to indicate more clearly that a  $\gamma_{ct}$ -set is an edge dominating set containing an edge cut, and has nothing to do with a possible operation of splitting edges.

The *edge connectivity* of a connected graph  $G$ , denoted  $\lambda(G)$ , equals the minimum cardinality of a set of edges  $F \subseteq E$  such that  $\langle V, E - F \rangle$  is disconnected. Such a set is called a  $\lambda$ -set, or a *minimum edge cut*. We note that for a connected graph  $G$ , both a  $\gamma'$ -set and  $\lambda$ -set exist. Thus, a  $\gamma_{ct}$ -set exists for any (connected) graph  $G$ .

## 2. VALUES AND BOUNDS

In this section we establish the value of the edge cut domination number for various classes of graphs, and we establish a variety of inequalities between this parameter and other known parameters of graphs. The following inequalities are consequences of the definitions of the given parameters, and are stated without proof.

**Proposition 2.1.** For any connected graph  $G$ ,

$$(i) \gamma'(G) \leq \gamma_{ct}(G) \quad \text{and} \quad (ii) \lambda(G) \leq \gamma_{ct}(G).$$

Most of the following statements were previously observed by Neeralagi and Nayak [4], although without proof or comment. Here we provide proofs of these statements.

**Proposition 2.2.**

- (i) For the complete graph  $K_n$  of order  $n$ ,  $\gamma_{ct}(K_n) = n - 1$ .
- (ii) For the cycle  $C_n$  of order  $n \geq 4$ ,  $\gamma_{ct}(C_n) = \lceil \frac{n}{3} \rceil$ .
- (iii) For the wheel  $W_n$  of order  $n + 1$ ,  $\gamma_{ct}(W_n) = \lceil \frac{n-4}{3} \rceil + 3$ .
- (iv) For the complete bipartite graph  $K_{m,n}$ , with  $m \geq n$ ,  $\gamma_{ct}(K_{m,n}) = n$ .
- (v) For any tree  $T$ ,  $\gamma_{ct}(T) = \gamma'(T)$ .
- (vi) For the path  $P_n$  of order  $n$ ,  $\gamma_{ct}(P_n) = \lceil \frac{n-1}{3} \rceil$ .

*Proof.* (i) We have  $\lambda(K_n) = n - 1 \leq \gamma_{ct}(K_n)$ . Note that the set of all edges incident to a given vertex is an edge cut dominating set of cardinality  $n - 1$ . Hence  $\gamma_{ct}(K_n) = n - 1$ .

(ii) For  $n \geq 4$ , it is clear that  $\lambda(C_n) = 2$ . Moreover,  $\gamma'(C_n) = \lceil \frac{n}{3} \rceil \geq 2$ . It follows that  $\gamma_{ct}(C_n) = \gamma'(C_n) = \lceil \frac{n}{3} \rceil$ .

(iii) For a given vertex of degree 3, select the three edges incident to this vertex so that the resulting subgraph is disconnected. Note that dominating the remaining edges is equivalent to dominating  $P_{n-3}$ . Since  $\gamma'(P_{n-3}) = \lceil \frac{n-4}{3} \rceil$ , we have an edge cut dominating set of cardinality  $\lceil \frac{n-4}{3} \rceil + 3$ . Moreover, such a set is a minimum cardinality edge cut dominating set.

(iv) The set of all edges incident to a given vertex in the partition of  $K_{m,n}$  with  $m$  vertices is an edge cut dominating set of cardinality  $n$ . Since  $\lambda(K_{m,n}) = n$ , it follows that  $\gamma_{ct}(K_{m,n}) = n$ .

(v) Note that every edge of  $T$  is a cut edge. Hence every  $\gamma'$ -set disconnects  $T$ . It follows that  $\gamma_{ct}(T) = \gamma'(T)$ .

(vi) Since  $P_n$  is a tree, we have  $\gamma_{ct}(T) = \gamma'(T) = \lceil \frac{n-1}{3} \rceil$ .

□

The *edge covering number*  $\alpha_1(G)$  is the minimum cardinality of a set  $F$  of edges such that every vertex is incident with at least one edge in  $F$ . The *matching number*  $\beta_1(G)$  is the maximum cardinality over all independent edge sets.

**Proposition 2.3.** For any connected graph  $G$  with size  $m > 1$ ,

$$\gamma_{ct}(G) \leq m - \beta_1(G).$$

*Proof.* Let  $F$  an independent set of edges in  $G$ , that is no two edges in  $F$  have a vertex in common. Now consider the complement of  $F$ ,  $E - F$ . Since  $F$  is an independent set of edges, we see that the removal of the edge set  $E - F$  from  $G$  disconnects  $G$ . Moreover,  $E - F$  is an edge dominating set. For if not, then there exists some edge  $e \in F$  such that  $e$  is not adjacent to any edge in  $E - F$ . But of course  $e$  is not adjacent to any edge in  $F$ . But this contradicts the fact  $G$  is a connected graph. Hence  $E - F$  is an edge cut dominating set. Hence  $\gamma'_{ct}(G) \leq |E - F| = m - |F|$ .

Since  $\beta_1(G)$  is the maximum cardinality over all independent edge sets, it follows that  $\gamma'_{ct}(G) \leq m - \beta_1(G)$ .  $\square$

The following corollary follows from the fact that for a connected graph  $G = (V, E)$ ,  $\alpha_1(G) + \beta_1(G) = |V|$ .

**Corollary 2.4.** For any tree  $T$  of order  $n$ ,

$$\gamma_{ct}(G) \leq n - \beta_1(G) - 1 = \alpha_1(G) - 1.$$

Given the above lower and upper bounds for  $\gamma_{ct}(G)$ , it is of interest to determine for which classes of graphs any of the three following expressions hold:

$$(i) \gamma'(G) = \gamma_{ct}(G), \quad (ii) \lambda(G) = \gamma_{ct}(G), \quad (iii) \gamma_{ct}(G) \leq \alpha_1(G) - 1.$$

Here we look briefly at some classes of graphs where  $\gamma'(G) = \gamma_{ct}(G)$ . There are of course some trivial cases. For example, if there exists a  $\gamma'$ -set which contains a cut edge of  $G$ , then  $\gamma'(G) = \gamma_{ct}(G)$ . Note however, that if  $\gamma'(G) = \gamma_{ct}(G)$  it is not necessarily the case that there exists a  $\gamma'$ -set containing a cut edge. For example,  $\gamma'(K_{m,n}) = \gamma_{ct}(K_{m,n})$ , but  $K_{m,n}$  contains no cut edge. There are of course an infinite number of graphs with the property that there exists a  $\gamma'$ -set containing a cut edge. It is clear that any tree has this property. More generally, if  $G$  is a connected graph containing three adjacent cut edges, then every  $\gamma'$ -set of  $G$  contains a cut edge. For in order to dominate  $G$ , an edge set must contain at least one of the three adjacent cut edges.

The following proposition presents another infinite class of graphs which have a  $\gamma'$ -set containing a cut edge.

**Proposition 2.5.** For a graph  $G$  defined by  $K_n$  and  $K_m$  ( $m, n > 2$ ) connected by a path of length 1 or 2,  $\gamma_{ct}(G) = \gamma'(G)$  if and only if  $m$  or  $n$  is even.

*Proof.* First recall that  $\gamma_{ct}(K_n) = n - 1$ , but  $\gamma'(K_n) = \lfloor n/2 \rfloor$ . So a  $\gamma_{ct}$ -set must contain a cut edge from the path connecting  $K_n$  and  $K_m$ .

Consider the case when  $K_m$  and  $K_n$  are connected by a single edge, say  $e$ . Note that  $e$  is a cut edge. Now there are two distinct edge sets to consider. Let  $E_1$  be an

edge dominating set of minimal cardinality which does not contain  $e$ . Let  $E_2$  be an edge dominating set of minimal cardinality which does contain  $e$ . Given that  $\gamma'(K_n) = \lfloor n/2 \rfloor$ , we see that

$$|E_1| = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \quad |E_2| = 1 + \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$$

Now we make the observation that  $\gamma'(G) = \min\{|E_1|, |E_2|\}$  and  $\gamma_{ct}(G) = |E_2|$ . Hence we have  $\gamma'(G) = \gamma_{ct}(G)$  if and only if  $|E_2| \leq |E_1|$ . It is straightforward to check that  $|E_2| \leq |E_1|$  if and only if at least one of  $m$  and  $n$  is even.

Next consider the case when  $K_m$  and  $K_n$  are connected by a path of length two. Let the two edges of the path be denoted by  $d$  and  $e$ , and say  $d$  is incident to a vertex in  $K_m$  and  $e$  is incident to a vertex in  $K_n$ . In this case there are four sets of edges to consider. Let  $E_1, E_2, E_3, E_4$  each be edge dominating sets of minimum cardinality which also satisfy the following conditions:

$$d, e \notin E_1, \quad d \in E_2, e \notin E_2, \quad d \notin E_3, e \in E_3, \quad d, e \in E_4.$$

Again using the observation that  $\gamma'(K_n) = \lfloor n/2 \rfloor$  we have that

$$|E_1| = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \quad |E_2| = 1 + \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$$

$$|E_3| = 1 + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor \quad |E_4| = 2 + \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$$

It is clear that  $\gamma'(G) = \min\{|E_1|, |E_2|, |E_3|, |E_4|\}$  and also that  $\gamma_{ct} = \min\{|E_2|, |E_3|, |E_4|\}$ . Hence  $\gamma'(G) = \gamma_{ct}(G)$  if and only if  $|E_i| \leq |E_1|$  for some  $i \in \{2, 3, 4\}$ . Again a straightforward check shows that this occurs if and only if at least one of  $m$  and  $n$  is even.  $\square$

### 3. INTRODUCTION OF NEW PARAMETERS

In the mid 1970s Cockayne and Hedetniemi [1, 2] noted the following chain of inequalities for any graph  $G$ :

$$\text{ir}(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq \text{IR}(G).$$

Here  $\Gamma(G)$ , the *upper domination number*, is the maximum cardinality taken over all minimal dominating sets of  $G$ . The *independent domination number* and *independence number*,  $i(G)$  and  $\beta(G)$ , are respectively the minimum and maximum cardinalities taken over all maximal sets of independent vertices of  $G$ . Similarly,  $\text{ir}(G)$  and  $\text{IR}(G)$ , the *lower* and *upper irredundance numbers*, are respectively the minimum and maximum cardinalities taken over all maximal irredundant sets of vertices of  $G$ .

The introduction and study of new parameters often involves a similar chain of inequalities as the one above. Indeed, this inequality chain has been instrumental in the research of many parameters. In what follows we establish such a chain relating parameters which we define corresponding to edge cut domination, edge cut irredundance, and edge cut independence.

**Definition 3.1.** An edge cut dominating set  $F$  is a *minimal edge cut dominating set* if for any edge  $e$  in  $F$  either

- (1)  $F - \{e\}$  is not an edge dominating set, or
- (2)  $F - \{e\}$  is not an edge cut.

**Definition 3.2.** Let  $G = (V, E)$  be a graph. Then

$$\Gamma_{ct}(G) = \max\{|F| : F \text{ is a minimal edge cut dominating set}\}.$$

Since a minimal edge cut dominating set is first and foremost an edge cut dominating set, it is clear that  $\gamma_{ct}(G) \leq \Gamma_{ct}(G)$ .

**Definition 3.3.** Let  $G = (V, E)$  be a graph and  $F \subseteq E$ . Then an edge  $e \in F$  has a *private neighbor* with respect to  $F$  if either

- (1)  $e$  is an independent edge in  $F$ , or
- (2)  $\exists e' \notin F$  such that  $e'$  is adjacent to  $e$  and no other edges of  $F$ .

**Definition 3.4.** Let  $G = (V, E)$  be a graph, and  $F \subseteq E$ . Then an edge  $e \in F$  is *irredundant* if  $e$  has a private neighbor with respect to  $F$ . If each edge in  $F$  is irredundant, then we say  $F$  is irredundant.

**Definition 3.5.** Let  $G = (V, E)$  be a graph. Then  $F \subseteq E$  is an *edge cut irredundant set* if for every edge  $e \in F$  either

- (1)  $e$  is irredundant, or
- (2)  $F - \{e\}$  is not an edge cut.

**Definition 3.6.** An edge cut irredundant set  $F$  is called *maximal* if  $F \cup \{e\}$  is not edge cut irredundant for every  $e \in E - F$ .

**Definition 3.7.** Let  $G = (V, E)$  be a graph. Then

- $\text{ir}_{ct} = \min\{|F| : F \text{ is a maximal edge cut irredundant set}\}$ , and
- $\text{IR}_{ct} = \max\{|F| : F \text{ is a maximal edge cut irredundant set}\}$ .

**Proposition 3.8.** A minimal edge cut dominating set  $F$  is a maximal edge cut irredundant set.

*Proof.* Let  $F$  is a minimal edge cut dominating set. Now for any edge  $e \in F$ , either  $F - \{e\}$  is not an edge dominating set, which means that  $e$  has a private neighbor with respect to  $F$ , i.e.,  $e$  is irredundant; or  $F - \{e\}$  is not an edge cut. So  $F$  is an edge cut irredundant set. Now let  $e \in E - F$  and consider  $F \cup \{e\}$ . Since  $F$  is an edge dominating set, we know that  $F \cup \{e\}$  is also an edge dominating set. But this implies that  $e$  has no private neighbor with respect to  $F \cup \{e\}$ . For  $e$  cannot be independent in  $F \subseteq F \cup \{e\}$ , and every edge adjacent to  $e$  must also be adjacent to some edge in  $F$ . It follows that  $e$  is not irredundant in  $F \cup \{e\}$ . Hence  $F \cup \{e\}$  is not an edge cut irredundant set for any  $e \in E - F$ . Thus  $F$  is a maximal edge cut irredundant set.  $\square$

Note: A maximal edge cut irredundant set is not necessarily an edge cut dominating set. This can be seen in the following figure.

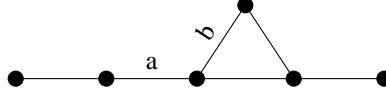


FIGURE 1. The edge set  $\{a, b\}$  is a maximal edge cut irredundant set but not an edge cut dominating set.

**Definition 3.9.** Let  $G = (V, E)$  be a graph, and  $F \subseteq E$ . Then  $F$  is an *edge cut independent set* if for every edge  $e \in F$  either

- (1)  $e$  is independent in  $F$ , or
- (2)  $F - \{e\}$  is not an edge cut.

**Definition 3.10.** An edge cut independent set  $F$  is called *maximal* if  $F \cup \{e\}$  is not an edge cut independent set for every edge  $e \in E - F$ .

**Definition 3.11.** Let  $G = (V, E)$  be a graph. Then

- $i_{ct}(G) = \min\{|F| : F \text{ is maximal edge cut independent set}\}$ , and
- $\beta_{ct}(G) = \max\{|F| : F \text{ is maximal edge cut independent set}\}$ .

**Proposition 3.12.** A maximal edge cut independent set is a minimal edge cut dominating set.

*Proof.* Let  $F$  be a maximal edge cut independent set. We first show that  $F$  is an edge dominating set. For suppose this is not the case. Then there exists some  $e \in E - F$  such that  $e$  is not adjacent to any edge in  $F$ . But this implies that  $e$  is independent in  $F \cup \{e\}$ , which contradicts the maximality of  $F$ . Hence  $F$  is an edge (cut) dominating set. Now we show that  $F$  is a minimal edge cut dominating set. Let  $e \in F$ , then either  $e$  is independent in  $F$ , or  $F - \{e\}$  is not an edge cut. If the latter is true, we are done. On the other hand, if  $e$  is independent in  $F$ , then  $e$  is not adjacent to any edge in  $F$ , which implies that  $F - \{e\}$  is not an edge dominating set. Therefore,  $F$  is a minimal edge cut dominating set.  $\square$

Note: A minimal edge cut dominating set is not necessarily a maximal edge cut independent set. This is shown in the following figure.

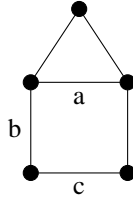


FIGURE 2. The edge set  $\{a, b, c\}$  is a minimal edge cut dominating set but not an edge cut independent set.

**Theorem 3.13.** For any connected graph  $G$ ,

$$\text{ir}_{ct}(G) \leq \gamma_{ct}(G) \leq i_{ct}(G) \leq \beta_{ct}(G) \leq \Gamma_{ct}(G) \leq \text{IR}_{ct}(G).$$

*Proof.* From Proposition 3.8 and the following note, we see that

$$\text{ir}_{ct}(G) \leq \gamma_{ct}(G) \leq \Gamma_{ct}(G) \leq \text{IR}_{ct}(G).$$

From Proposition 3.12 and the following note, we also see that

$$\gamma_{ct}(G) \leq i_{ct}(G) \leq \beta_{ct}(G) \leq \Gamma_{ct}(G).$$

Hence the desired inequality chain holds.  $\square$

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